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# Some Properties of the Principle ideal graph of the ring Z_p

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**Abstract:** In this study, definition anew graph of a ring z_p .A graph Principle ideal of the ring z_p denote by PIG(z_p) is present, where the graph's vertices stand in for ring z_p elements s.t any two vertices a and b merage by an edge if and only if ⟨a⟩=⟨b⟩ , where ⟨a⟩ is ring generated by a . In this paper we study some topological indices of PIG(z_p) . The topological indices use in chemical science.

**Keywords:** Commutative ring , Sum connectivity index, Forgotten index, 1st Zagreb index, The atomic bond connectivity index, Graph theory.

## 1. Introduction

Graph theory has become a very famous and rapidly developing field of mathematics due to its extensive theoretical advances and diverse applications to real-world problems. Although graph theory is still a relatively new field of research, it has produced many profound and novel discoveries in the past 20 years. Graphs can be used to represent a variety of relationships and processes in biological, social, physical, and information systems..

Sylvester introduced the term graph in an 1878 article in Nature Publishing [1]. The 1st book in graph theory was published by Denes in 1936 [2]. In the last decade of the 19th century, algebraic graph theory began to develop rapidly, and a large number of research articles were published in this branch of graph theory. The fascinating field of algebraic graph theory studies how algebra and graph theory interact. Algebraic methods can be used to prove graph theory facts in surprising and elegant ways. There are many interesting algebraic objects associated with graphs. In recent years, the study of algebraic graph theory has become increasingly important. In algebraic graph theory, properties of graphs are converted into algebraic properties and then theorems about graphs are derived from algebraic results and techniques.

However, many algebraic topics can also be understood by converting them into graphs and exploiting the properties of graphs. In (1999), Anderson and Livingston [3] assigned a simple graph to each commutative ring R and examined the interaction between the ring theoretic conditions and Graph properties of. Akbari et al. Bhavanari et al. (2010) merged. [3] rings R (not necessarily commutative) and defined a new concept of "primary graph of R" (denoted by PG(R)). They gave some examples and obtained some fundamentally important results about PG(R). (2013) Chelvam and Asir. [5] studied the advantages of ensemble graphs of commutative rings. In (2014) Patra and Kalita [6] studied prime graphs of commutative rings. In (2023) Nermen J. Khalel and Nabeel E. Arif [7] studied the association graphs of commutative rings.

In the articles , we study more topological metrics that need degrees of examples: eccentricity index [8], connectivity index [9], sum connectivity index [10], 1st and 2nd index [11], forgetting indicator [12], exponential geometric arithmetic [13], atomic bond connectivity index [13] and harmonic index [13]. Topological indexing of symmetry group graphs was introduced in (2019) Abdussakir [14], also examined in (2020) G. R. Roshini

[15], and also in (2020) G. R. Roshini [16] , Alaa .J and Akram .S [17] in (2021) study topological indices and (Schultz and Hosoya) polynomials of the intersection graph of subgroup of the group $Zr$.

**2.Primary Results:**

Definition 2.1(22): Let R a ring. a graph where G(V,E) where V(G)=R $/\{\mathbf{0}\}$ and E(G)=$\{\overline{\boldsymbol{ab}}\ ,\langle \boldsymbol{a}\rangle = \langle \boldsymbol{b}\rangle\ \ \boldsymbol{and}\ \boldsymbol{a} \neq \boldsymbol{b}\}$ is called the principle ideal graph of R indicate by PIG (R)

In this article , all diagrams are simplfy, bounded, connect, and unidirectional. For the graph $G$ =($V$($G$), $E$($G$)) let $d$($u$) be the degree of vertex $u$ in $G$. If $d$($u$) = 0, then u is an isolated vertex. Let $d$ ($u$, $v$) be the distance between two distinct vertices u and v.

The $ecc(u)$ of the vertex $u$ is $ecc(u)$ = $sup$ $\{d(u, v): v \in V(G)\}$ Next definitions refers to a topological indices of a graph $G$ Eccentricity Index of $G$ is [8] $\boldsymbol{\xi^c}$ $(G)))$ =$\sum_{\boldsymbol{u}\in \boldsymbol{V}(\mathbf{G})} \mathbf{d}(\mathbf{u}).\mathbf{e}(\mathbf{u})$

An Eccentricity connectivity index of $G$ is[9].

$$\boldsymbol{X}(\boldsymbol{G}) = \sum_{\boldsymbol{uv\epsilon E}((\boldsymbol{G})} \frac{\mathbf{1}}{\sqrt{\mathbf{d}(\boldsymbol{u})\,.\,\mathbf{d}\,(\boldsymbol{v})}}$$

A Sum connectivity index of $G$ is [10]

$$\boldsymbol{S}(\boldsymbol{G}) = \sum_{\boldsymbol{uv\epsilon E}((\boldsymbol{G})} \frac{\mathbf{1}}{\sqrt{\mathbf{d}(\boldsymbol{u}) + \mathbf{d}\,(\boldsymbol{v})}}$$

A 1st zagreb index of G is [11]

$$\boldsymbol{M_1}(\boldsymbol{G}) = \sum_{\boldsymbol{u}\in \boldsymbol{V}(\boldsymbol{G})} (\mathbf{d}(\boldsymbol{u}))^2$$

A 2nd zagreb index of $G$ is [11]

$$\boldsymbol{M_2}(\boldsymbol{G}) = \sum_{\boldsymbol{uv}\in \boldsymbol{E}(\boldsymbol{G})} \mathbf{d}(\boldsymbol{u})\,.\,\mathbf{d}\,(\boldsymbol{v})$$

A forgotten index of $G$ is [12]

$$\boldsymbol{f}(\boldsymbol{G}) = \sum_{\boldsymbol{u}\in \boldsymbol{V}(\boldsymbol{G})} (\mathbf{d}(\,\boldsymbol{u}))^3$$

The atomic bond connectivity index of G is [13]

$$\boldsymbol{Abc}(\boldsymbol{G}) = \sum_{\boldsymbol{uv}\in \boldsymbol{E}(\boldsymbol{G})} \sqrt{\frac{\mathbf{d}(\boldsymbol{u}) + \mathbf{d}(\boldsymbol{v}) - \mathbf{2}}{\mathbf{d}(\boldsymbol{u})\,.\,\mathbf{d}\,(\boldsymbol{v})}}$$

The Geometric arithmetic index of $G$

is [13]

$$GA(G) = \sum_{uv\in E(G))} \frac{2\sqrt{d(u).d(v)}}{d(u)+d(v)}$$

Harmonic index of $G$ is [13]

$$H(G) = \sum_{uv\in E(G))} \frac{2}{d(u)+d(v)}$$

Remark3.1: Let $PIG(R)$ be a graph of $R$ with $R = Z_p$ . Then .

$d$ (1) = $d$ (p-1) = $p-2$.

A distance between any two distinct vertices a and b is $d(a,b) \leq 2$ .

corollary3.2: Let $Z_p$ be a ring , $p$ is prime number , then $PIG(Z_p)$ is a complete graph $K_p$

Corollary3.3: If $Z_p$ **be a ring** , $p$ is a prime number , then $PIG(Z_p)$ is a Regular graph.

*3.Some Topological indices of $PIG(Z_p)$*

*NOTE: For every $p \geq 3$ , p is prime number then:*

$d(n) = p-2$ , $n = 1, \ldots, p-1$

$e(n) = 1$ , $n = 1, \ldots, p-1$

*Theorem3.1: The ecc connectivity index of $NZG(Z_p)$ is*

$\xi((PIG(Z_p)) = p^2 - 2p$

*Proof:*

$$\xi(PIG(Z_p)) = \sum_{u\in V(PIG(Z_p))} d(u).e(u)$$

$$= \underbrace{e(1)\,d(1) \ldots \ldots + e(p-1)d\,(p-1)}_{p\ times}$$

$$= p^2 - 2p$$

*Theorem3.2: The connectivity*

*Index of* $PIG(Z_p)$ *is* $X(PIG(Z_p)) = \frac{\sum_{i=2}^{p-1} p-i}{p-2}$

*Proof:*

$$X(PIG(Z_p)) = \sum_{uv\epsilon E(X(PIG(Z_p))} \frac{1}{\sqrt{d(u).d(v)}}$$

$$= \underbrace{\frac{1}{\sqrt{d(0).d\,(1)}} + \cdots + \frac{1}{\sqrt{d(0).d\,(p-1)}}}_{(p-1)times} + \underbrace{\frac{1}{\sqrt{d(1).d\,(2)}} + \cdots + \frac{1}{\sqrt{d(1).d\,(p-1)}}}_{(p-2)times}$$

$$+ \cdots + \frac{1}{\sqrt{d(p-2).d\,(p-1)}}$$

$$= \frac{\sum_{i=2}^{p-1} p-i}{p-2}$$

*Theorem3.3: : The Sum connectivity index of* $\boldsymbol{PIG(Z_p)}$ *is*

$$S(PIG(Z_p) = \frac{\sum_{i=2}^{p-1} p-i}{\sqrt{2p-4}}$$

*Proof:*

$$S(PIG(Z_p) = \sum_{uv\epsilon E(PIG(Z_p)} \frac{1}{\sqrt{d(u)+d\,(v)}}$$

$$= \underbrace{\frac{1}{\sqrt{d(0)+d\,(1)}} + \cdots + \frac{1}{\sqrt{d(0)+d\,(p-1)}}}_{(p-1)times}$$

$$+ \underbrace{\frac{1}{\sqrt{d(1)+d\,(2)}} + \cdots + \frac{1}{\sqrt{d(1)+d\,(p-1)}}}_{(p-2)times} + \cdots + \frac{1}{\sqrt{d(p-2)+d\,(p-1)}}$$

$$= \frac{\sum_{i=2}^{p-1} p-i}{\sqrt{2p-4}}$$

*Theorem3.4: The 1st Zagreb index of* $\boldsymbol{PIG(Z_p)}$ *is*

$$M_1(PIG(Z_p) = p\,(p-2)^2$$

*Proof:*

$$M_1(PIG(Z_p) = \sum_{u\in V(PIG(Z_p)} (d(u))^2$$

$$= \underbrace{(d(1))^2 + \cdots + (d(p-1))^2}_{p\ times}$$

$$= p\,(p-2)^2$$

*Theorem3.5: The second Zagreb index of* $\boldsymbol{PIG(Z_p)}$ *is*

$$M_2(PIG(Z_p) = \sum_{i=1}^{p-1}(p-i)\,(p-2)^2$$

*Proof:*

$$M_2(PIG(Z_p) = \sum_{uv\in E(PIG(Z_p)} d(u).d\,(v)$$

$$= \underbrace{d(1).d(2) + \cdots + d(1).d\,(p-1)}_{(p-1)times} + \underbrace{d(2).d(3) + \cdots + d(2).d\,(p-1)}_{(p-2)times} + \cdots$$

$$+ d(p-1).d\,(p-2)$$

$$= \sum_{i=2}^{p-1} (p-i).(p-1)^2$$

*Theorem3.6: The forgotten index of* $PIG(Z_p)$ *is*

$$F(PIG(Z_p) = p\,(p-2)^3$$

*Proof:*

$$F(NZG(Z_p) = \sum_{u \in V(NZG(Z_p)} (d(u))^3$$

$$= \underbrace{(d(1))^3 + \cdots + (d(p-1))^3}_{p\ times}$$

$$= p\,(p-2)^3$$

*Theorem3.7: The Atom bond connectivity index of* $PIG(Z_p)$ *is*

$$ABC(PIG(Z_p) = \frac{\sum_{i=2}^{p-1}(p-i)}{p-2} . \sqrt{2p-6}$$

*Proof:*

$$ABC(PIG(Z_p) = \sum_{uv \in E(PIG(Z_p)} \sqrt{\frac{d(u) + d(v) - 2}{d(u).d\,(v)}}$$

$$= \underbrace{\sqrt{\frac{d(1) + d(2) - 2}{d(1).d\,(2)}} + \cdots + \sqrt{\frac{d(1) + d(p-1) - 2}{d(1).d\,(p-1)}}}_{(p-1)times}$$

$$+ \underbrace{\sqrt{\frac{d(2) + d(3) - 2}{d(2).d(3)}} + \cdots + \sqrt{\frac{d(2) + d(p-1) - 2}{d(2).d(p-1)}}}_{(p-2)times} + \cdots +$$

$$\sqrt{\frac{d(p-1) + d(p-2) - 2}{d\,(p-1).d\,(p-2)}}$$

$$= \frac{\sum_{i=1}^{p-1}(p-i)}{p-2} . \sqrt{2p-6}$$

*Theorem3.8: The Geometric-Arithemtic index of* $PIG(Z_p)$ *is*

$$GA(PIG(Z_p) = \frac{\sum_{i=2}^{p-1}(p-i)}{2p-4} . (2p\text{-}4)$$

*Proof:*

$$GA(PIG(Z_p) = \sum_{uv \in E(PIG(Z_p))} \frac{2\sqrt{d\,(u).d\,(v)}}{d\,(u) + d\,(v)}$$

$$= \underbrace{\frac{2\sqrt{d\,(1).d\,(2)}}{d\,(1) + d\,(2)} +..+ \frac{2\sqrt{d\,(1).d\,(p-1)}}{d\,(1) + d\,(p-1)}}_{(p-1)times}$$

$$+ \underbrace{\frac{2\sqrt{d\,(2).d\,(3)}}{d\,(2) + d\,(3)} + \cdots + \frac{2\sqrt{d\,(2).d\,(p-1)}}{d\,(2) + d\,(p-1)}}_{(p-3)times}$$

$$+ \cdots + \frac{2\sqrt{d(p-1).d(p-2)}}{d(p-1) + d(p-2)}$$

$$= \frac{\sum_{i=2}^{p-1}(p-i)}{2p-2} \cdot (2p\text{-}4)$$

*Theorem3.9: The Harmonic index of* $PIG(Z_p)$ *is*

$$H(PIG(Z_p) = \frac{\sum_{i=2}^{p-1}(p-i)}{2p-2}$$

*Proof:*

$$H(PIG(Z_p) = \sum_{uv\in E(PIG(Z_p))} \frac{2}{d\,(u) + d\,(v)}$$

$$= \underbrace{\frac{2}{d\,(1) + d\,(2)} + \cdots + \frac{2}{d\,(1) + d\,(p-1)}}_{(p-1)times}$$

$$+ \underbrace{\frac{2}{d\,(2) + d\,(3)} + \cdots + \frac{2}{d\,(2) + d\,(p-1)}}_{(p-2)times}$$

$$+ \cdots + \frac{2}{d(p-1) + d\,(p-2)}$$

$$= \frac{\sum_{i=2}^{p-1}(p-i)}{2p-4}$$

## *4.Conclusions*

In this study, formulas for some degrees and eccentricity based topological indices are proposed for principle ideal graph of ring $Z$p, where p is a prime number. In the futherTo investigate further, examine the graph of the ring $Zpq$, where $p$, $q$ are prime numbers

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